



RKN SCHEME WITH CONTINUOUS COEFFICIENTS FOR THE APPROXIMATION OF SECOND ORDER ORDINARY INITIAL VALUE PROBLEMS (IVPS)



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Abstract: The derivation of implicit Runge-Kutta-Nystrom (RKN) scheme with continuous coefficients for the direct approximation of special second Order ordinary initial value problems(IVPs) using the theory of s-stage Runge-Kutta (RK) for first order ordinary differential equations is presented. The study provides the use of both collocation and interpolation procedure to obtain the scheme. Based on a homogeneous test model, the stability and error analysis of the scheme is also investigated. The continuous formulation of the integrator will enable us to evaluate at some grid and off grid points in the integration interval. The advantage of the continuous scheme as against the discrete schemes for the direct integration of the second order special IVP includes the fact that multiple discrete schemes can be obtained. Numerical examples have been included to demonstrate the accuracy of the scheme.

Keywords: Numerical analysis, ordinary differential equations, initial value problems

Introduction

In this paper, we proposed that using the well-known properties of the s-stage implicit RK methods for first order ordinary differential equation; it is possible to obtain an implicit RKN method with continuous coefficients for the numerical approximation of the special second order IVPs.

$$y'' = f(x, y), y(x_0) = y_0, y'(x_0) = y'_0 \quad (1.1)$$

Having periodic solution; such problems often arises in different areas of engineering applications and applied

sciences such as celestial mechanics, seismology and electrodynamics. The general s-stage RKN method for the direct integration of the general second order IVPs

$$y'' = f(x, y, y'), y(x_0) = y_0, y'(x_0) = y'_0 \quad (1.2)$$

Where the primes denote differentiation with respect to x and $f: \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined in the form;

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{j=1}^s b_j f_j, y'_{n+1} = y'_n + h \sum_{j=1}^s b_j f_j \quad (1.3)$$

$$f_j = f(x_n + c_j h, y_n + c_j h y'_n + h^2 \sum_{k=1}^s a_{jk} f_k, y'_n + h \sum_{k=1}^s \bar{a}_{jk} f_k)$$

Where: s denote the number of stages of the method, $h = x_{n+1} - x_n$ is the step size and y_{n+1}, y'_{n+1} are approximations to the exact solution $y(x_{n+1})$ and its derivative $y'(x_{n+1})$ respectively. The RKN parameters a_{jk}, b_j, b'_j and c_j defines the method and the c_j 's satisfy the row simplifying assumption;

$$\frac{1}{2} c_j^2 = \sum_{k=1}^s a_{jk} \quad (1.4)$$

When the right-hand side of the system (1.2) is independent of y' , the RKN simplifies to a special RKN method. This is defined, see for example sharp *et al.* (1990), Franco and Gomez (2009), Imoni *et al.* (2014) by;

Where;

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{j=1}^s b_j f_j, y'_{n+1} = y'_n + h \sum_{j=1}^s b_j f_j \quad (1.5)$$

$$f_j = f(x_n + c_j h, y_n + c_j h y'_n + h^2 \sum_{k=1}^s a_{jk} f_k)$$

The coefficients of the method are compactly represented by means of the Butcher-array;

c	A
	b^T
	b'^T

Where;

$$c = [c_1, \dots, c_s]^T, [b_1, \dots, b_s]^T, b'_1 = [b'_1, \dots, b'_s]^T$$

and $A = [a_{ij}]$ with $C \in \mathbb{R}, b^T, b'^T \in \mathbb{R}^s$ and $A \in \mathbb{R}^{s \times s}$

Implicit Runge-Kutta methods for the first order differential equations

The general s-stage RK methods, Butcher (2008), Hairer *et al.* (1993), Lambert (1991) for any IVPs

$$y' = f(x, y), y(x_0) = y_0, f: \mathbb{R}^N \rightarrow \mathbb{R}^N \quad (1.6)$$

is defined by;

$$\begin{array}{c|c}
 c & \bar{A} : A \\
 \hline
 & b^T : b'^T
 \end{array}$$

Where;

$$\left. \begin{aligned}
 y_{n+1} &= y_n + h \sum_{i=1}^s b_i k_i \\
 k_i &= f \left(x_n + c_i h, y_n + h \sum_{j=1}^s a_{ij} k_j \right), \quad i=1, 2, \dots, s
 \end{aligned} \right\} \quad (1.7)$$

It is also assume that the row-sum condition holds:

$$c_i = \sum_{j=1}^s a_{ij}, \quad i = 1, 2, \dots, s, \quad \sum_{i=1}^s b_i = 1 \quad (1.8)$$

It is convenient to represent the coefficients of RK methods in the Butcher tableau;

$$\begin{array}{c|c}
 c & A \\
 \hline
 & b^T
 \end{array}
 \equiv
 \begin{array}{c|c}
 c_1 & \alpha_{11} \alpha_{12} \dots \alpha_{1s} \\
 c_2 & \alpha_{21} \alpha_{22} \dots \alpha_{2s} \\
 \vdots & \vdots \quad \vdots \quad \dots \quad \vdots \\
 c & \alpha_{s1} \alpha_{s2} \dots \alpha_{ss} \\
 \hline
 & w_1 \quad w_2 \dots w_s
 \end{array}$$

with

$$c = [c_1, c_2, \dots, c_s]^T, w = [w_1, w_2, \dots, w_s]^T, A = [\alpha_{ij}]$$

With implicit RK methods, a general non-linear system of equations must be solved at every time step for all stages simultaneously. However, because of their superiority stability properties, implicit RK methods are well-suited for periodic and stiff problems (Iserles, 1996)

As earlier mentioned in section 1.1, the general second order system (1.2), considering the vectors (y, y') as new variables is given by

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ f(x, y, y') \end{pmatrix}, \quad \begin{matrix} y(x_0) = y_0 \\ y'(x_0) = y'_0 \end{matrix} \quad (1.9)$$

See Hairer *et al.* (1993). However, Hairer and Wanner (1976) established that for the coefficients $\bar{A} = \bar{a}_{jk}$ in (1.4), the same conditions are obtained as for classical RK methods. Hence, the coefficients of any RK method (of order p) can be taken as \bar{A} in (1.4). Then put $A = \bar{A}^2$, $c = \bar{A}e$, then all order conditions (up to order p) are satisfied, since this choice corresponds exactly to the RK method \bar{A} applied to the system (1.9). Apply method (1.2) to (1.9) to obtain the methods (1.3) and (1.5) when applied to (1.1) and (1.2), respectively with $A = (\bar{a}_{ij}) = \beta^2$,

$$\bar{A} = (\bar{a}_{ij}) = \beta, \beta = \beta e, c = \bar{A}e, b^T = W, b'^T = W^T \beta$$

.This is symbolize in the Butcher form as;

See Jain *et al.* (1989), Adeboye and Shaba (2011)

Some attempts have been made to solve the general second order IVPs (1.2) directly without reduction to a first-order system of equations using collocation approach. For example, Adeboye and Shaba (2011) proposed a collocation method for the solution of the general second IVPs (1.2) directly without reducing it to first order systems of ODEs. In a recent paper of Adeboye and Shaba (2011), the authors demonstrated an application of LMMs to solve directly the general second order IVPs (1.2) by using collocation approach. In the case of the special second order IVP (1.1), little or nothing seemed to have been done; in fact we are not aware of any of such paper. The aim of this paper is to demonstrate using the theory of the s-stage RK methods for first order ODEs, RKN method can be obtained by collocation procedure for the direct integration of the special IVPs (1.1). This work extends the formalism of Jain *et al.* (1989) and Adeboye and Shaba (2011) to the special second order IVPs (1.2)

Construction of the RKN scheme with continuous coefficients

Consider the power series:

$$p(x) = \sum_{j=0}^{\infty} a_j x^j \quad (2.1)$$

This is used as the basis or trial function to produce an approximate solution to (1.1), (1.2) and (1.6) as;

$$y(x) = \sum_{j=0}^{t+m-1} a_j x^j \quad (2.2)$$

where

$$a_j \in \mathbb{R}, j = 0(1)t + m - 1, y \in C^m(a, b) \subset P(x)$$

The a_j are the parameters to be determined, t and m are number of points of interpolation and collocation points. Expressed (2.2) as

$$y(x) = \left(\sum_{j=0}^{t-1} \varphi_{j,t+m-1} y_{n+j} + \sum_{j=0}^{m-1} h \varphi_{j,t+m-1} f_{n+j} \right) (1, x, x^2, \dots, x^{t+m-1})^T \quad (2.3a)$$

and can be written explicitly as

$$y(x) = [y_n \dots y_{n+t-1}, f_n \dots f_{n+m-1}] C^T \begin{bmatrix} 1 \\ x \\ x^2 \\ \dots \\ x^{t+m-1} \end{bmatrix} \quad (2.3b)$$

$$C = \begin{bmatrix} \varphi_{0,1} & \dots & \varphi_{t-1,1} & h\varphi_{0,1} & \dots & h\varphi_{m-1,1} \\ \varphi_{0,2} & \dots & \varphi_{t-1,2} & h\varphi_{0,2} & \dots & h\varphi_{m-1,2} \\ \varphi_{0,t} & \dots & \varphi_{t-1,t} & h\varphi_{0,t} & \dots & h\varphi_{m-1,t} \end{bmatrix} = D^{-1} \quad (2.3c)$$

of dimension $(t + m) \times (t + m)$ and

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \dots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \dots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \dots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2\bar{x}_n & \dots & (t+m-1)\bar{x}_n^{(t+m-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2\bar{x}_n^{m-1} & \dots & (t+m-1)\bar{x}_n^{(t+m-2)} \end{bmatrix} \quad (2.3d)$$

$$y'(x) = \sum_{j=0}^{m+1-1} ja_j x^{j-1} = f(x, y) \quad (2.4)$$

$$y''(x) = \sum_{j=0}^{m+1-1} j(j-1)a_j x^{j-2} = f(x, y, y') \quad (2.5)$$

Interpolate (2.2) at the grid points $x_{n+j}, j = 0, 1, \dots, t-1$ and collocate (2.3) at $x_{n+j}, j = 0, 1, \dots, m-1$ chosen from the given step $[x_n, x_{n+k}]$. In this case $k = 5, m = t = 3$

and both at points $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$. Using the general multistep

collocation methods (see Yahaya and Adeghoye (2007), lead to the following;

This matrix is called the collocation matrix and entries of C are the constant coefficients of the polynomials given in (2.2) which assume an approximate solution to (1.1), (1.2) and (1.6) in the form (2.2). The first and second derivatives of (2.2) are respectively

$$D = \begin{pmatrix} 1 & x_n + \frac{h}{5} & (x_n + \frac{h}{5})^2 & (x_n + \frac{h}{5})^3 & (x_n + \frac{h}{5})^4 & (x_n + \frac{h}{5})^5 \\ 1 & x_n + \frac{2h}{5} & (x_n + \frac{2h}{5})^2 & (x_n + \frac{2h}{5})^3 & (x_n + \frac{2h}{5})^4 & (x_n + \frac{2h}{5})^5 \\ 1 & x_n + \frac{3h}{5} & (x_n + \frac{3h}{5})^2 & (x_n + \frac{3h}{5})^3 & (x_n + \frac{3h}{5})^4 & (x_n + \frac{3h}{5})^5 \\ 0 & 1 & 2x_n + \frac{4h}{5} & 3(x_n + \frac{h}{5})^2 & 4(x_n + \frac{h}{5})^3 & 5(x_n + \frac{h}{5})^4 \\ 0 & 1 & 2x_n + \frac{4h}{5} & 3(x_n + \frac{2h}{5})^2 & 4(x_n + \frac{2h}{5})^3 & 5(x_n + \frac{2h}{5})^4 \\ 0 & 1 & 2x_n + \frac{6h}{5} & 3(x_n + \frac{3h}{5})^2 & 4(x_n + \frac{3h}{5})^3 & 5(x_n + \frac{3h}{5})^4 \end{pmatrix} \quad (2.6)$$

With the help of maple 13 mathematical software, we inverted matrix D, substituted into (2.3d) and obtain the continuous form;

$$y(x) = \left(\frac{-(72h^5 + 1140x_n h^4 + 6350x_n^2 h^3 + 16375x_n^3 h^2 + 20000x_n^4 h + 9375x_n^5) + E}{4h^5} \right) y_{n+\frac{1}{5}} + \left(\frac{(9h^4 + 120x_n h^3 + 550x_n^2 h^2 + 100x_n^3 h + 625x_n^4) - 20(6h^3 + 55x_n h^2 + 150x_n^2 h + 125x_n^3)x + F}{h^4} \right) y_{n+\frac{2}{5}} + \left(\frac{8h^5 + 132x_n h^4 + 830x_n^2 h^3 + 2475x_n^3 h^2 + 3500x_n^4 h + 1875x_n^5 + G}{4h^5} \right) y_{n+\frac{3}{5}} + \left(\frac{-(36h^5 + 480x_n h^4 + 2425x_n^2 h^3 + 5875x_n^3 h^2 + 6875x_n^4 h + 3125x_n^5) + H}{20h^4} \right) f_{n+\frac{1}{5}} + \left(\frac{-(18h^5 + 285x_n h^4 + 1700x_n^2 h^3 + 4750x_n^3 h^2 + 6250x_n^4 h + 3125x_n^5) + I}{5h^4} \right) f_{n+\frac{2}{5}} + \left(\frac{-(12h^5 + 200x_n h^4 + 1275x_n^2 h^3 + 3875x_n^3 h^2 + 5625x_n^4 h + 3125x_n^5) + J}{20h^5} \right) f_{n+\frac{3}{5}} \quad (2.7)$$

where

$$\begin{aligned}
 E &= 5(228h^4 + 2540x_n h^3 + 9825x_n^2 h^2 + 16000x_n^3 h + 9375x_n^4)x - 25(254h^3 + 1965x_n h^2 + \\
 &\quad 4800x_n^2 h + 3750x_n^3)x^2 + 125(131h^2 + 640x_n h + 750x_n^2)x^3 - 625(32h + 75x_n)x^4 + 9375x^5, \\
 F &= 50(11h^2 + 60x_n h + 75x_n^2)x^2 - 500(2h + 5x_n)x^3 + 625x^4, \\
 G &= -5(132h^4 + 1660x_n h^3 + 7525x_n^2 h^2 + 14000x_n^3 h + 9375x_n^4)x + 25(166h^3 + 1485x_n h^2 + \\
 &\quad 4200x_n^2 h + 3750x_n^3)x^2 - 125(99h^2 + 560x_n h + 750x_n^2)x^3 + 625(28h + 75x_n)x^4 - 9375x^5, \\
 H &= 5(96h^4 + 970x_n h^3 + 3525x_n^2 h^2 + 5500x_n^3 h + 3125x_n^4)x - 25(97h^3 + 705x_n h^2 + \\
 &\quad 1650x_n^2 h + 1250x_n^3)x^2 + 125(47h^2 + 220x_n h + 250x_n^2)x^3 - 625(11h + 25x_n)x^4 + 3125x^5, \\
 I &= (57h^4 + 680x_n h^3 + 2850x_n^2 h^2 + 5000x_n^3 h + 3125x_n^4)x - 10(34h^3 + 285x_n h^2 + \\
 &\quad 750x_n^2 h + 625x_n^3)x^2 + 50(19h^2 + 100x_n h + 125x_n^2)x^3 - 625(2h + 5x_n)x^4 + 625x^5, \\
 J &= 25(8h^4 + 102x_n h^3 + 465x_n^2 h^2 + 900x_n^3 h + 625x_n^4)x - 25(51h^3 + 465x_n h^2 + \\
 &\quad 1350x_n^2 h + 1250x_n^3)x^2 + 125(31h^2 + 180x_n h + 250x_n^2)x^3 - 625(9h + 25x_n)x^4 + 3125x^5
 \end{aligned}$$

Evaluating the continuous formula in equation (2.7) and its first derivative at points a $x = x_n$ and $x = x_{n+1}$, results in the following block hybrid schemes

$$\begin{aligned}
 18y_{n+\frac{1}{5}} - 9y_{n+\frac{2}{5}} - 10y_{n+\frac{3}{5}} + y_n &= \frac{h}{5} \left(-9f_{n+\frac{1}{5}} - 18f_{n+\frac{2}{5}} - 3f_{n+\frac{3}{5}} \right) \\
 -117y_{n+\frac{1}{5}} - 64y_{n+\frac{2}{5}} + 180y_{n+\frac{3}{5}} + y_{n+1} &= \frac{h}{5} \left(36f_{n+\frac{1}{5}} + 192f_{n+\frac{2}{5}} + 72f_{n+\frac{3}{5}} \right) \\
 -285y_{n+\frac{1}{5}} + 120y_{n+\frac{2}{5}} + 165y_{n+\frac{3}{5}} &= h \left(24f_{n+\frac{1}{5}} + 57f_{n+\frac{2}{5}} + 10f_{n+\frac{3}{5}} - f_n \right) \\
 -1110y_{n+\frac{1}{5}} - 480y_{n+\frac{2}{5}} + 1590y_{n+\frac{3}{5}} &= h \left(69f_{n+\frac{1}{5}} + 352f_{n+\frac{2}{5}} + 120f_{n+\frac{3}{5}} - f_{n+1} \right)
 \end{aligned} \tag{2.8}$$

Then solve for $y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+1}$ in the block hybrid scheme (2.8) simultaneously and obtain the following block scheme:

$$\begin{aligned}
 y_{n+\frac{1}{5}} &= y_n + \frac{h}{18000} (1275f_n + 3135f_{n+\frac{1}{5}} - 1130f_{n+\frac{2}{5}} + 340f_{n+\frac{3}{5}} - 19f_{n+1}) \\
 y_{n+\frac{2}{5}} &= y_n + \frac{h}{2250} (146f_n + 615f_{n+\frac{1}{5}} + 130f_{n+\frac{2}{5}} + 10f_{n+\frac{3}{5}} - f_{n+1}) \\
 y_{n+\frac{3}{5}} &= y_n + \frac{h}{2000} (138f_n + 495f_{n+\frac{1}{5}} + 390f_{n+\frac{2}{5}} + 180f_{n+\frac{3}{5}} - 3f_{n+1}) \\
 y_{n+1} &= y_n + \frac{h}{144} (2f_n + 75f_{n+\frac{1}{5}} - 50f_{n+\frac{2}{5}} + 100f_{n+\frac{3}{5}} + 17f_{n+1})
 \end{aligned} \tag{2.9}$$

Each of the above schemes has order five and error constant $\left(\frac{173}{112500000}, \frac{1}{1171875}, \frac{21}{12500000}, -\frac{1}{60000} \right)^T$, respectively.

The coefficients as characterized by Butcher array (2.1), are obtain, respectively as

Table 1: The coefficients for the first derivative

0	0	0	0	0	0
1	$\frac{637}{5}$	$\frac{209}{5}$	$-\frac{113}{5}$	$\frac{17}{5}$	$-\frac{19}{5}$
2	$\frac{73}{5}$	$\frac{41}{5}$	$\frac{13}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$
3	$\frac{69}{5}$	$\frac{99}{5}$	$\frac{39}{5}$	$\frac{9}{5}$	$-\frac{3}{5}$
1	$\frac{1}{72}$	$\frac{25}{48}$	$-\frac{25}{72}$	$\frac{25}{36}$	$\frac{17}{144}$
	$\frac{1}{72}$	$\frac{25}{48}$	$-\frac{25}{72}$	$\frac{25}{36}$	$\frac{17}{144}$

Table 2: The coefficients for the implicit RKN method (1.1) and (1.2)

0	0	0	0	0	0	0	0	0	0	
1	$\frac{637}{5}$	$\frac{209}{5}$	$\frac{113}{5}$	$\frac{17}{5}$	$\frac{19}{5}$	$\frac{2147}{225000}$	$\frac{173}{10000}$	$\frac{473}{45000}$	$\frac{179}{45000}$	$\frac{139}{45000}$
2	$\frac{73}{5}$	$\frac{41}{5}$	$\frac{13}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{658}{28125}$	$\frac{241}{3750}$	$\frac{8}{625}$	$\frac{31}{5625}$	$\frac{7}{18750}$
3	$\frac{69}{5}$	$\frac{99}{5}$	$\frac{39}{5}$	$\frac{9}{5}$	$\frac{3}{5}$	$\frac{909}{25000}$	$\frac{1179}{10000}$	$\frac{69}{5000}$	$\frac{63}{5000}$	$\frac{33}{50000}$
1	$\frac{1}{72}$	$\frac{25}{48}$	$\frac{25}{72}$	$\frac{17}{144}$		$\frac{23}{360}$	$\frac{11}{48}$	$\frac{1}{24}$	$\frac{11}{72}$	$\frac{1}{80}$
	$\frac{1}{72}$	$\frac{25}{48}$	$\frac{25}{72}$	$\frac{17}{144}$		$\frac{23}{360}$	$\frac{11}{48}$	$\frac{1}{24}$	$\frac{11}{72}$	$\frac{1}{80}$

Substituting all the parameters into the special form of RKN method (1.5) we obtain the implicit RKN scheme which is suitable for the integration of second order initial value problem possessing oscillatory solution as,

$$\begin{aligned}
 y_{n+1} &= y_n + hy'_n + h^2 \left(\frac{23}{360} f_1 + \frac{11}{48} f_2 + \frac{1}{24} f_3 + \frac{11}{72} f_4 + \frac{1}{80} f_5 \right) \\
 y'_{n+1} &= y'_n + h \left(\frac{1}{72} f_1 + \frac{25}{48} f_2 - \frac{25}{72} f_3 + \frac{25}{36} f_4 + \frac{17}{144} f_5 \right)
 \end{aligned}
 \tag{2.10}$$

where

$$\begin{aligned}
 f_1 &= f(x_n, y_n) \\
 f_2 &= f\left(x_n + \frac{1}{5}h, y_n + \frac{1}{5}hy'_n + h^2 \left(\frac{2147}{225000} f_1 + \frac{173}{10000} f_2 - \frac{473}{45000} f_3 + \frac{179}{45000} f_4 - \frac{139}{45000} f_5 \right)\right) \\
 f_3 &= f\left(x_n + \frac{2}{5}h, y_n + \frac{2}{5}hy'_n + h^2 \left(\frac{658}{28125} f_1 + \frac{241}{3750} f_2 - \frac{8}{625} f_3 + \frac{31}{5625} f_4 - \frac{7}{18750} f_5 \right)\right) \\
 f_4 &= f\left(x_n + \frac{3}{5}h, y_n + \frac{3}{5}hy'_n + h^2 \left(\frac{909}{25000} f_1 + \frac{1179}{10000} f_2 + \frac{69}{5000} f_3 + \frac{63}{5000} f_4 - \frac{33}{50000} f_5 \right)\right) \\
 f_5 &= f(x_n + h, y_n + hy'_n + h^2 \left(\frac{23}{360} f_1 + \frac{11}{48} f_2 + \frac{1}{24} f_3 + \frac{11}{72} f_4 + \frac{1}{80} f_5 \right))
 \end{aligned}$$

The interval of stability of the method is also investigated. Thus the amplification matrix obtained is,

$$R(z) = \begin{bmatrix} 1 + \frac{1}{2}z + \frac{13}{50}z^2 + \frac{7}{250}z^3 + \frac{1}{100}z^4 + \frac{99}{5000}z^5 & 1 + \frac{4}{25}z + \frac{1}{125}z^2 + \frac{1}{500}z^3 + \frac{3}{10000}z^4 - \frac{89}{100000}z^5 \\ z + \frac{13}{26}z^2 + \frac{49}{500}z^3 + \frac{3}{500}z^4 + \frac{3}{20000}z^5 & 1 + \frac{1}{2}z + \frac{21}{500}z^2 + \frac{1}{500}z^3 + \frac{13}{5000}z^4 - \frac{47}{312500}z^5 \end{bmatrix}
 \tag{2.11}$$

The stability region plotted using MATLAB software shows that the implicit RKN method has the stability interval of approximately (6.83, 0). The stability plot is depicted in Fig. 1.

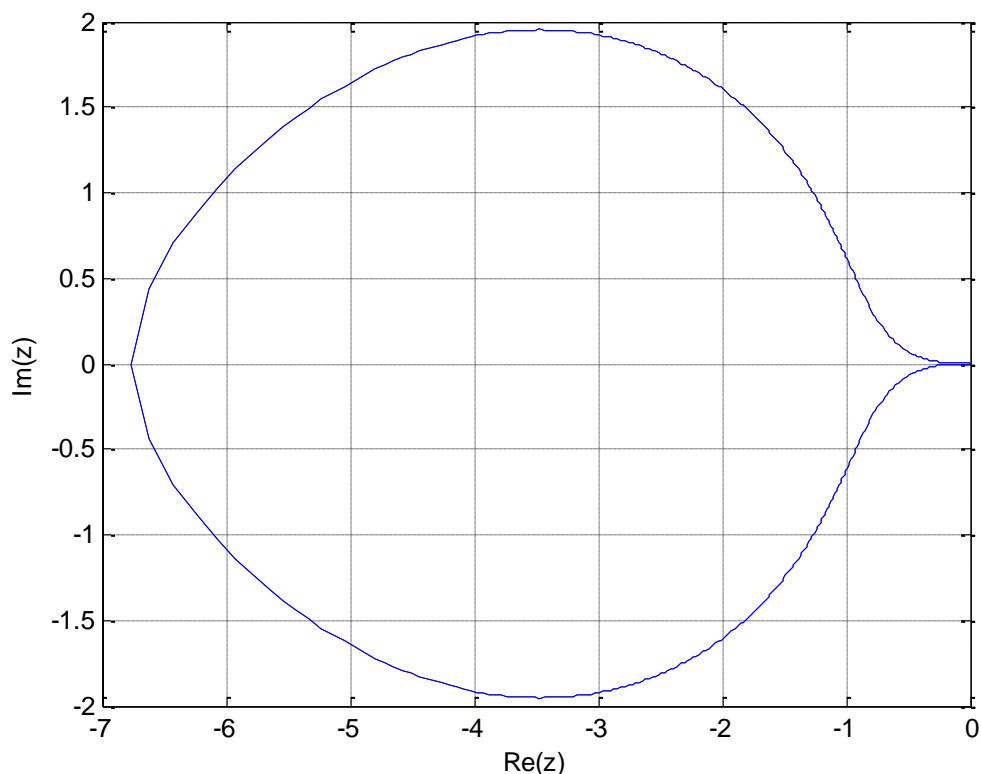


Fig. 1: Stability region for the implicit RKN method of Table 2

Numerical examples

In this section, the new method is applied to two well-known periodic initial value problems. These problems have exact solution, thus their actual error is compared. In tables 1-2, the following notations are used.

E_{max}: maximum error, h: step size, Steps: the number of steps

FCN: the number of functions evaluation

The maximum error is defined as the absolute value of the computed solution minus the exact solution.

$$E_{max} = \max(\|y_n - (x_n)\|)$$

Problem 1: Consider the inhomogeneous equation

$$y'' = 99 \sin x - 100y, y(0) = 1, y'(0) = 11$$

The exact solution of the problem is given as $y(x) = \cos(10x) + \sin(10x + \sin(x)), 0 \leq x \leq 10\pi$

Source: Papageorgiou and Famelis (2001)

Problem 2: We next consider the second order initial value problem

$$y'' = -\omega^2 y + (\omega^2 - 1)\sin x; y(0) = 1, y'(0) = 1 + \omega; 0 \leq x \leq 2$$

with exact solution

$$y(x) = \cos(\omega x) + \sin(\omega x) + \sin(x), \omega \geq 1, \quad (\text{Here}$$

we take $\omega = 10$)

Sources: Simos (1993)

Table 1: Absolute Maximum errors for Problem 1

h	Steps	FCN	E _{max}
0.10	356	5762	1.58273841E-01
0.02	3141	15705	2.55013946E-02
0.001	58534	39265	2.25804076E-03
0.005	628423	202548	3.68678628E-03
0.0005	8283167	3141554	3.64915346E-03

Table 2: Absolute maximum errors for Problem 2

h	Steps	FCN	E _{max}
0.10	314	1570	1.10502516E-02
0.02	1570	7850	2.25602032E-02
0.001	7853	39265	4.35236343E-03
0.005	62562	525667	5.65779254E-03
0.0005	62831	314155	7.854915346E-3

From the numerical results in Tables 1 and 2, it can be seen that the new method compared favorably with the exact solution of the special second order IVPs considered with oscillatory behavior in terms of global error. Hence, the method is suitable for the integration of the IVPs (1.1).

Conclusion

We have been able to establish an implicit Runge-Kutta-Nystrom scheme with continuous coefficients for the direct numerical approximation of the special second order initial value problems (IVPs) in ordinary differential equations. The derived method is shown to have non-vanishing intervals of periodicity and we also give the error constants. Numerical

results obtained from the two problems solved converge with the exact solutions for various values of step length. This approach seems to us very promising for analysis and derivation of new numerical methods for ODEs. The extension to high order methods with larger stability interval and their implementation might be an interesting option for future research.

Conflict of Interest

Author declares that there is no conflict of interest reported in this paper.

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